Multi-terminal network power measurement

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Abstract This paper presents a simple and general theory on power absorbed by multi-terminal networks and the way to measure it, which has direct application to unification of active and reactive power measurement procedures of polyphase loads. This approach to the subject is highly efficient, not only for its simplicity, but also because it eliminates all particular demonstrations or proofs of different measurement methods, with consequent time and effort savings for students. It also offers absolute certainty about the scope of every power measurement procedure and their possible variations.

Keywords active power; multi-terminal network; power measurement; reactive power; wattmeter

Different power measurement procedures applied to polyphase loads have historically been developed separately, normally to give single solutions to specific problems[1,2] and extending the arrangement for monophase loads to three-phase systems.[3,4] Text books and manuals still present this subject with particular case-by-case demonstrations.[5,6] This form of presentation consequently needs to justify the adequacy of the proposed measurement methods for each type of three-phase load (wye, delta, balanced) and the same for other polyphase loads. In addition to assimilation difficulty for students, this approach generates great uncertainty with respect to possible or impossible wattmeter connections, depending on the actual load configuration. For example, the student knows that, in order to measure the power of a Y-connected three-phase load with neutral, he can use the three wattmeters method as shown in Fig. 1(a). But he might ignore whether the configuration of connections of Fig. 1(b), where one of the wattmeters is placed on the neutral and some monophase, wye and delta loads are connected, will offer the power absorbed by the whole set. The student may also know that two wattmeters connected as shown in

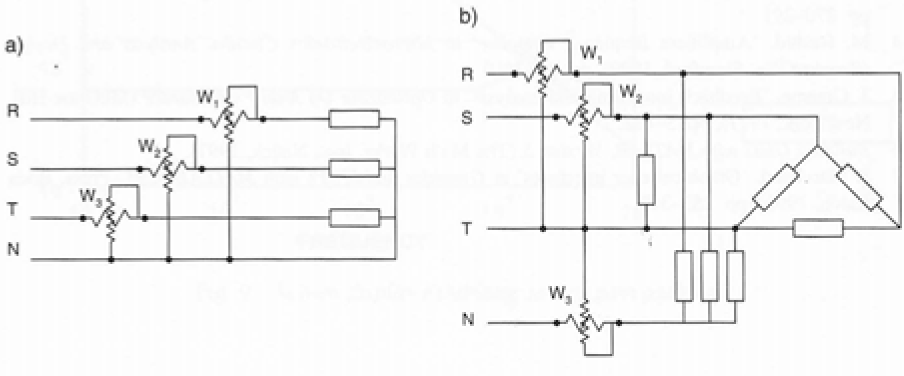

Fig. 1 *Configurations of connections to measure the power of a three-phase load.*

Fig. 10, with their respective current coils on the phases, can measure the two-phase load power, but he might not know if the same procedure is applicable when a Z_3 extra impedance is added to the former load (Fig. 12), or when a wattmeter is placed on the neutral instead of on one of the phases (Fig. 11(a)). In general, the information the student receives contributes to his uncertainty on matters such as these.

As will be proven, we can arrive at a unified and very easy-to-follow demonstration valid for all the cases of polyphase receivers and, in fact, for every multi-terminal network, whatever its excitation may be. The following demonstration would eradicate the possibility of common doubts. From the concept of multi-terminal network power, every possible connection of wattmeters for power measurement is shown at once and with minimum effort.

Multi-terminal network power

Definition: *The power absorbed by a multi-terminal network is the sum of the powers absorbed by all its branches.*

$$p = \sum_{h=1}^{r} u_h j_h \tag{1}$$

The h branches are the internal branches. u_h and j_h are the voltage and current of the h internal branch with the appropriate directions. r is the number of internal branches of the multi-terminal network. p is the instantaneous power. Calculating the average value of the two members of expression (1) it is immediately deduced that the average power absorbed by the multi-terminal network is the sum of the average powers absorbed by all its branches (Fig. 2).

Multi-terminal network power theorem: *Let $v_{k(k=1,2,\ldots,t)}$ be the potentials, with respect to any arbitrary common point O, of the t terminals of a multi-terminal network,*

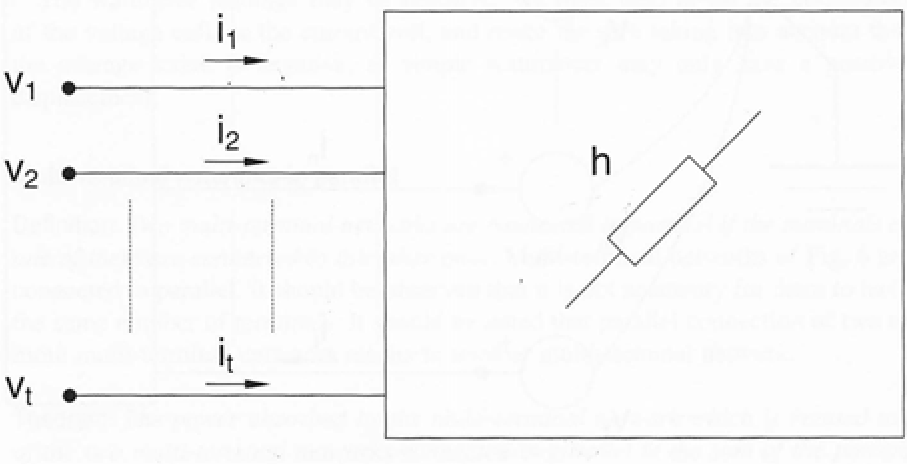

Fig. 2 *A multi-terminal network and one of its internal branches.*

and $i_{k(k=1,2,\ldots,t)}$ *the current entering the terminal k. The power absorbed by the multi-terminal network is*

$$p=\sum_{k=1}^{t} v_k i_k \qquad (2)$$

Demonstration: Let us consider a t-terminal network in which we suppose t voltage sources of value $v_{k(k=1,2,\ldots,t)}$ connected between every terminal and any arbitrary common point O. This means that each terminal has a v_k potential and that an i_k current enters it (Fig. 3). The Tellegen theorem[7,8] applied to the network formed in this way gives the expression

$$-v_1 i_1 - v_2 i_2 - \ldots - v_t i_t + p = 0 \qquad (3)$$

where $p=\sum_{h=1}^{r} u_h j_h \qquad (1)$

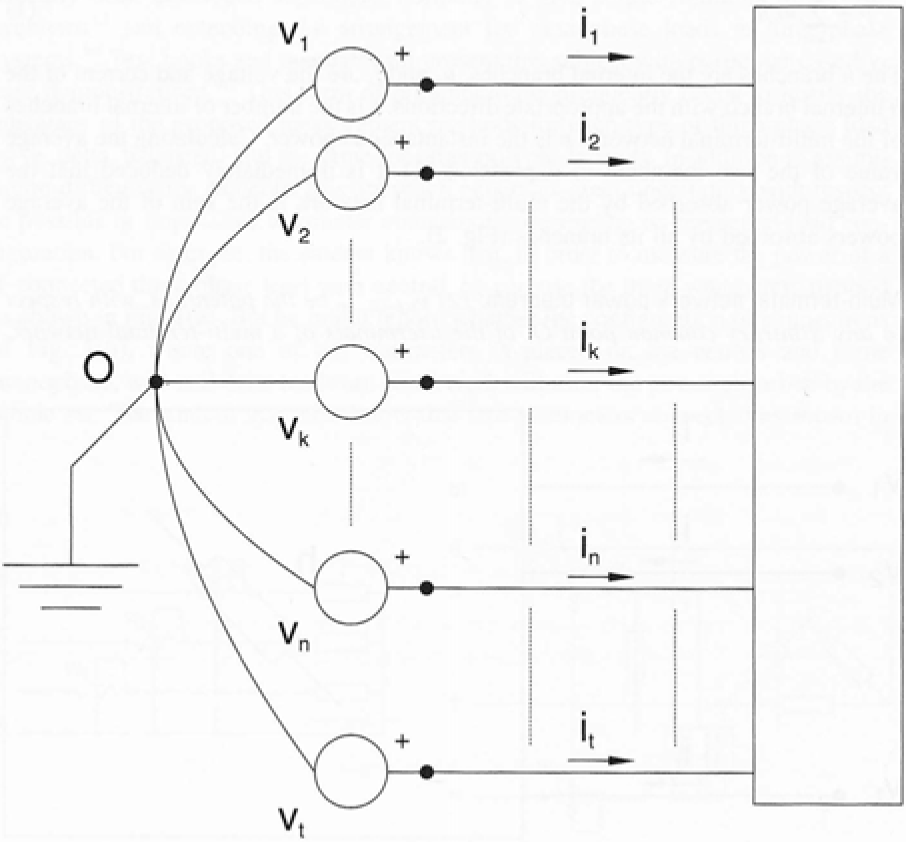

Fig. 3 *Each terminal has a* v_k *potential and an* i_k *current enters it.*

is the sum of the uj products of the internal branches of the multi-terminal network; we obtain

$$-\sum_{k=1}^{t} v_k i_k + p = 0 \tag{4}$$

and

$$p = \sum_{k=1}^{t} v_k i_k \tag{2}$$

which is the result we wanted to demonstrate.

As the reference point for terminal potentials is arbitrary, it may be chosen so as to coincide with one of the terminals of the multi-terminal network, i.e. the t terminal. The power absorbed by the multi-terminal network may then be written as

$$p = \sum_{k=1}^{n} v_k i_k \tag{5}$$

Each v_k voltage now designates the voltage between each k terminal and the t reference terminal; $n = t - 1$.

This is a theorem of considerable practical importance, as it procures the method to measure with $n = t - 1$ wattmeters the power of any t-terminal network. It is only necessary to connect the current coil in series with its respective terminal and the voltage coil between this terminal and the reference one; the sum of the readings of all the wattmeters is the power the multi-terminal network absorbs.

Obtaining the average value in both members we arrive at the fact that the average power value is the sum of the average values of the $v_k i_{k(k=1,2,\ldots,n)}$ products. But these average values are the wattmeter readings if products vary fast enough. Therefore, in order to measure the average power value of a multi-terminal network it is enough to connect the wattmeters as shown in Figs 4 or 5 and sum their indications.

The wattmeter readings may be negative. We must then invert the connection of the voltage coil, or the current coil, and make the sum taking into account that the average value is negative, as simple wattmeters may only have a positive displacement.

Multi-terminal networks in parallel

Definition: *Two multi-terminal networks are connected in parallel if the terminals of one of them are connected to the other ones.* Multi-terminal networks of Fig. 6 are connected in parallel. It should be observed that it is not necessary for them to have the same number of terminals. It should be noted that parallel connection of two or more multi-terminal networks results in another multi-terminal network.

Theorem: *The power absorbed by the multi-terminal network which is created out of the two multi-terminal networks connected in parallel is the sum of the powers absorbed by both of them.*

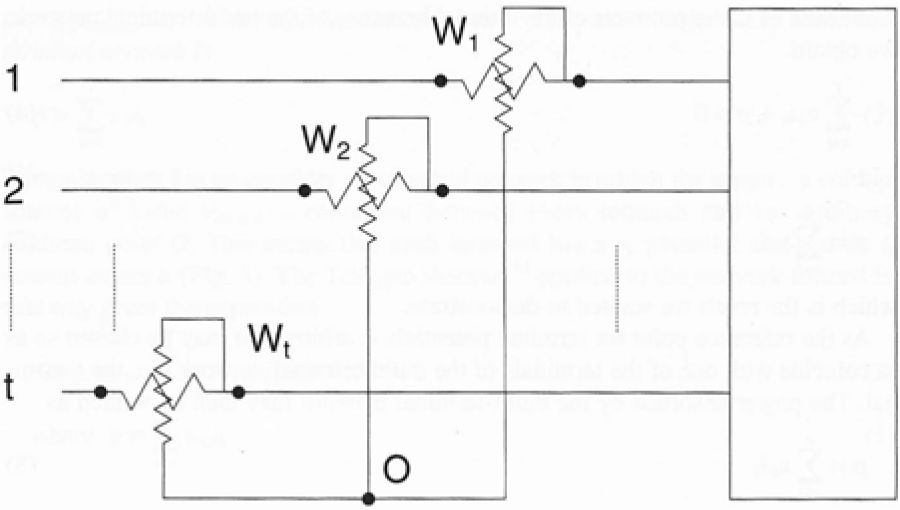

Fig. 4 *Configuration of connections to measure the power absorbed by a* t*-terminal network with* t *wattmeters. The power is the sum of all the readings of the wattmeters.*

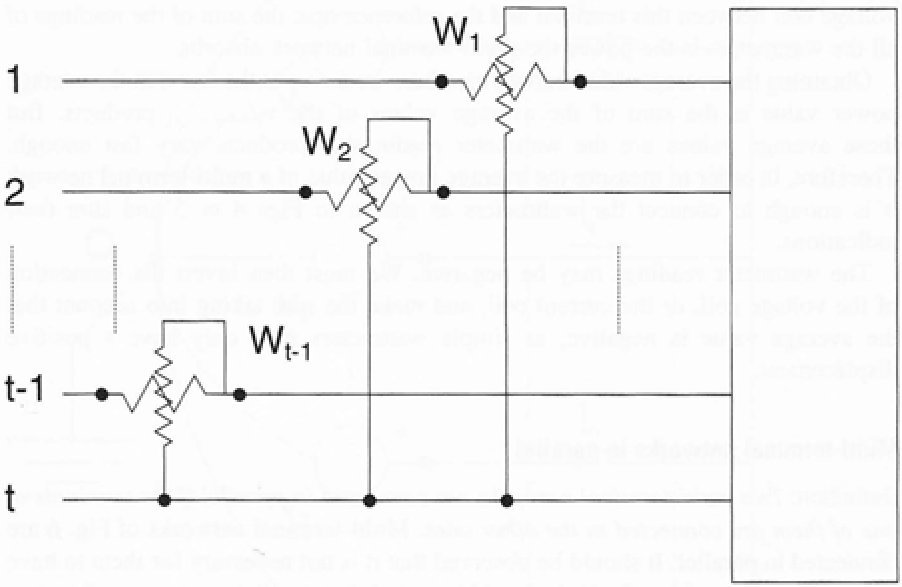

Fig. 5 *Configuration of connections to measure the power of a* t*-terminal network with* n = t − *1 wattmeters.*

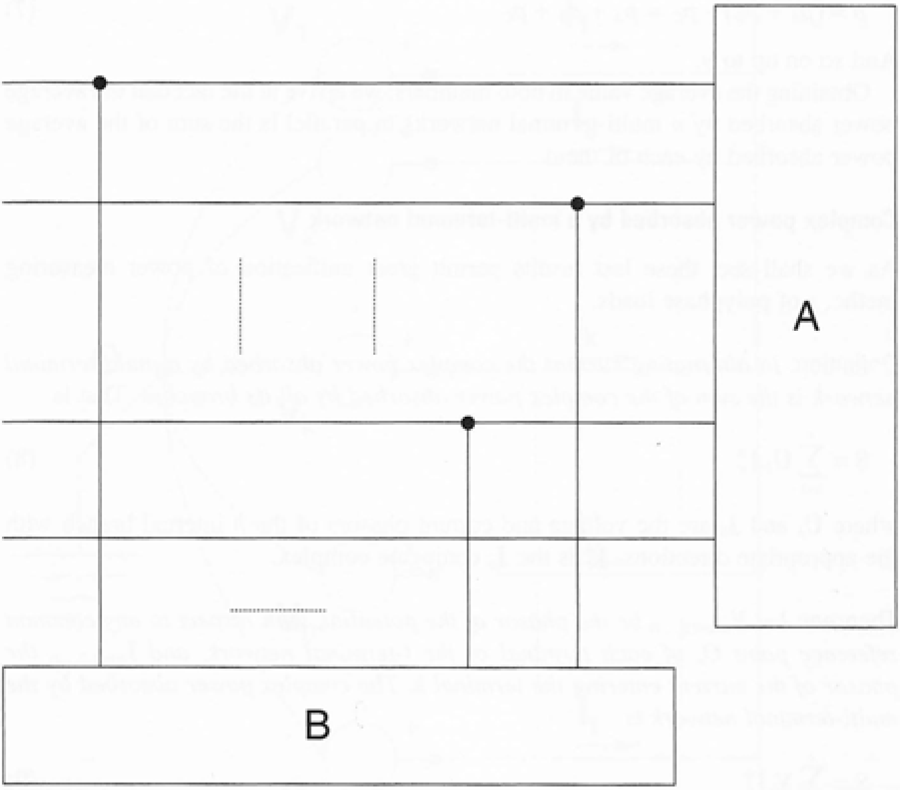

Fig. 6 *Multi-terminal networks in parallel.*

Demonstration: The power absorbed by the created multi-terminal network out of two multi-terminal networks A and B connected in parallel in Fig. 6 is the sum of the powers absorbed by all their branches:

$$p = \sum_{h_A=1}^{r_A} u_{h_A} j_{h_A} + \sum_{h_B=1}^{r_B} u_{h_B} j_{h_B} = p_A + p_B \quad (6)$$

as we wanted to demonstrate.

Corollary: *The power absorbed by n multi-terminal networks in parallel is the sum of the power absorbed by all of them.*

Demonstration: In effect, the connection of a third multi-terminal network C is equivalent to the connection of two multi-terminal networks: one resulting from the former two multi-terminal networks A and B, with the third C. So the power absorbed by this new created one is the sum of the power absorbed by C, plus the power absorbed by the one created from A and B. That is,

$$p = (p_A + p_B) + p_C = p_A + p_B + p_C \tag{7}$$

And so on up to n.

Obtaining the average value in both members, we arrive at the fact that the average power absorbed by n multi-terminal networks in parallel is the sum of the average power absorbed by each of them.

Complex power absorbed by a multi-terminal network

As we shall see, these last results permit great unification of power measuring methods of polyphase loads.

Definition: *In alternating current the complex power absorbed by a multi-terminal network is the sum of the complex power absorbed by all its branches.* That is

$$\mathbf{S} = \sum_{h=1}^{r} \mathbf{U}_h \mathbf{J}_h^* \tag{8}$$

where $\mathbf{U}_h$ and $\mathbf{J}_h$ are the voltage and current phasors of the h internal branch with the appropriate directions. $\mathbf{J}_h^*$ is the $\mathbf{J}_h$ conjugate complex.

Theorem: *Let $\mathbf{V}_{k(k=1,2,\ldots,t)}$ be the phasor of the potential, with respect to any common reference point O, of each terminal of the t-terminal network, and $\mathbf{I}_{k(k=1,2,\ldots,t)}$ the phasor of the current entering the terminal k. The complex power absorbed by the multi-terminal network is*

$$\mathbf{S} = \sum_{k=1}^{t} \mathbf{V}_k \mathbf{I}_k^* \tag{9}$$

Demonstration: Let us consider a t-terminal network in which we suppose that some voltage sources of value $\mathbf{V}_{k(k=1,2,\ldots,t)}$ are connected between each k terminal and the common reference point. In other words, every terminal has a $\mathbf{V}_k$ potential and an $\mathbf{I}_k$ current enters it. Boucherot's theorem affirms that the sum of the complex powers of all the branches of a network is zero.[8] When applied to the network formed as described above and illustrated in Fig. 7, Boucherot's theorem gives the following:

$$-\mathbf{V}_1\mathbf{I}_1^* - \mathbf{V}_2\mathbf{I}_2^* - \ldots - \mathbf{V}_t\mathbf{I}_t^* + \mathbf{S} = 0 \tag{10}$$

$$\text{where } \mathbf{S} = \sum_{h=1}^{r} \mathbf{U}_h \mathbf{J}_h^* \tag{11}$$

is the sum of the complex powers of the t-terminal network internal branches; that is, the complex power absorbed by the t-terminal network. Therefore we arrive at

$$\mathbf{S} = \mathbf{V}_1\mathbf{I}_1^* + \mathbf{V}_2\mathbf{I}_2^* + \ldots + \mathbf{V}_t\mathbf{I}_t^* \tag{12}$$

$$\text{or } \mathbf{S} = \sum_{k=1}^{t} \mathbf{V}_k \mathbf{I}_k^* \tag{9}$$

the result which we wanted to demonstrate.

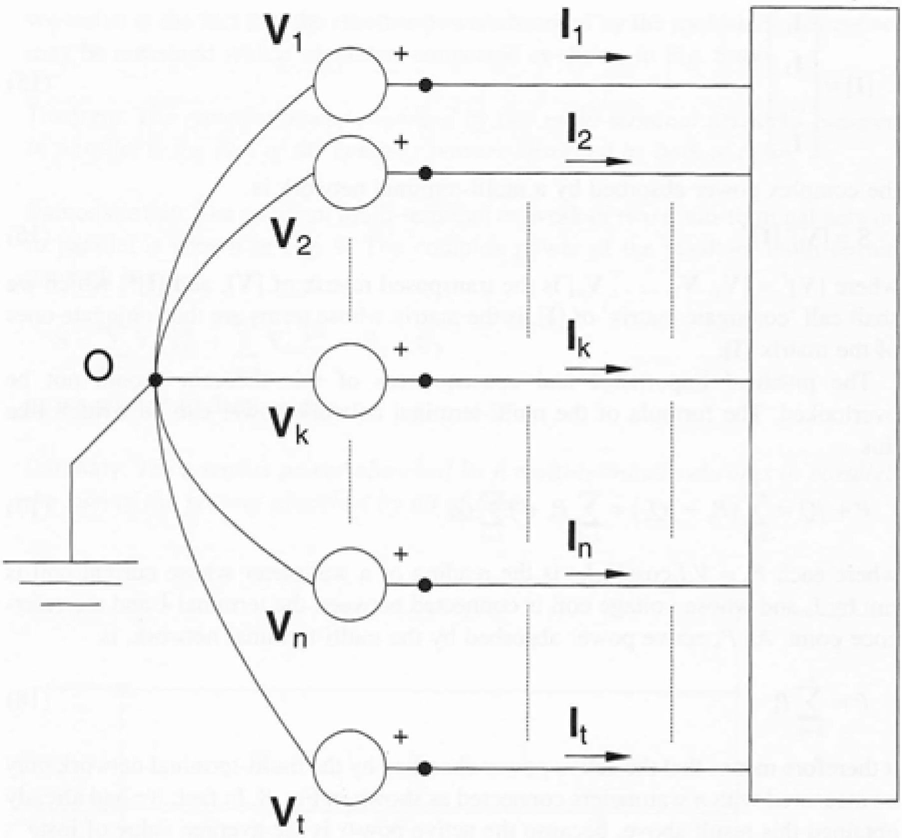

Fig. 7 *Each terminal has a* V_k *potential and* I_k *current enters it.*

One of the terminals is usually taken as reference point, i.e. the terminal *t*. The complex power of the *t*-terminal network is, then,

$$\mathbf{S} = \sum_{k=1}^{n} \mathbf{V}_k \mathbf{I}_k^* \tag{13}$$

where now $\mathbf{V}_k$ is the potential phasor of each terminal with respect to the terminal *t*, and $n = t - 1$.

If [**V**] is the matrix of the potential phasors of the *t* terminals, with respect to any common reference point *O*,

$$[\mathbf{V}] = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \cdots \\ \mathbf{V}_t \end{bmatrix} \tag{14}$$

and [**I**] the matrix of the current phasors entering each terminal,

$$[\mathbf{I}] = \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \dots \\ \mathbf{I}_t \end{bmatrix} \tag{15}$$

the complex power absorbed by a multi-terminal network is

$$\mathbf{S} = [\mathbf{V}]^T[\mathbf{I}]^* \tag{16}$$

where $[\mathbf{V}]^T = [\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n]$ is the transposed matrix of $[\mathbf{V}]$, and $[\mathbf{I}]^*$, which we shall call 'conjugate matrix' of $[\mathbf{I}]$, is the matrix whose terms are the conjugate ones of the matrix $[\mathbf{I}]$.

The practical importance and consequences of this theorem should not be overlooked. The formula of the multi-terminal network power can be written like this

$$P + jQ = \sum_{k=1}^{t}(P_k + jQ_k) = \sum_{k=1}^{t} P_k + j\sum_{k=1}^{t} Q_k \tag{17}$$

where each $P_k = V_k I_k \cos(V_k, I_k)$ is the reading of a wattmeter whose current coil is run by I_k and whose voltage coil is connected between the terminal k and the reference point. As P, active power absorbed by the multi-terminal network, is

$$P = \sum_{k=1}^{t} P_k \tag{18}$$

it therefore means that the active power absorbed by the multi-terminal network may be measured with n wattmeters connected as shown in Fig. 8. In fact, we had already obtained this result above, because the active power is the average value of instantaneous power. Similarly, $Q_k = V_k I_k \sin(\mathbf{V}_k, \mathbf{I}_k)$ is the reading of a varmeter whose current coil is run by $\mathbf{I}_k$ and whose voltage coil is connected between the terminal k and the reference one. As Q reactive power absorbed by the multi-terminal network is

$$Q = \sum_{k=1}^{n} Q_k \tag{19}$$

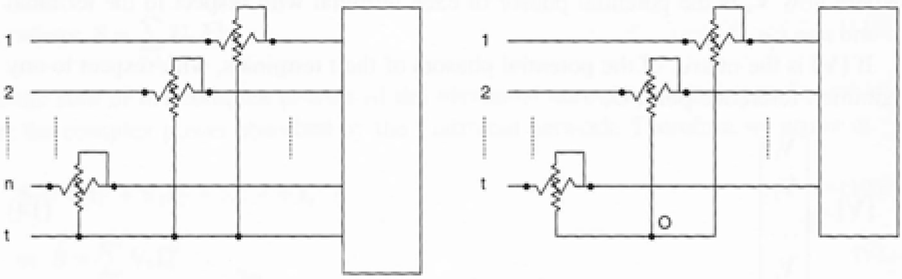

Fig. 8 *Connections for* n *and* t *varmeters.*

we arrive at the fact that the reactive power absorbed by the multi-terminal network may be measured with n varmeters connected as shown in Fig. 8 too.

Theorem: *The complex power absorbed by two multi-terminal networks connected in parallel is the sum of the complex powers absorbed by both of them.*

Demonstration: The resultant multi-terminal network of two multi-terminal networks in parallel is shown in Fig. 9. The complex power of the resultant multi-terminal network is

$$\mathbf{S} = \sum_{h_A=1}^{r_A} \mathbf{V}_{h_A} \mathbf{I}_{h_A}^* + \sum_{h_B=1}^{r_B} \mathbf{V}_{h_B} \mathbf{I}_{h_B}^* = \mathbf{S}_A + \mathbf{S}_B \tag{20}$$

as we wanted to demonstrate.

Corollary: *The complex power absorbed by n multi-terminal networks in parallel is the sum of the powers absorbed by all of them.*

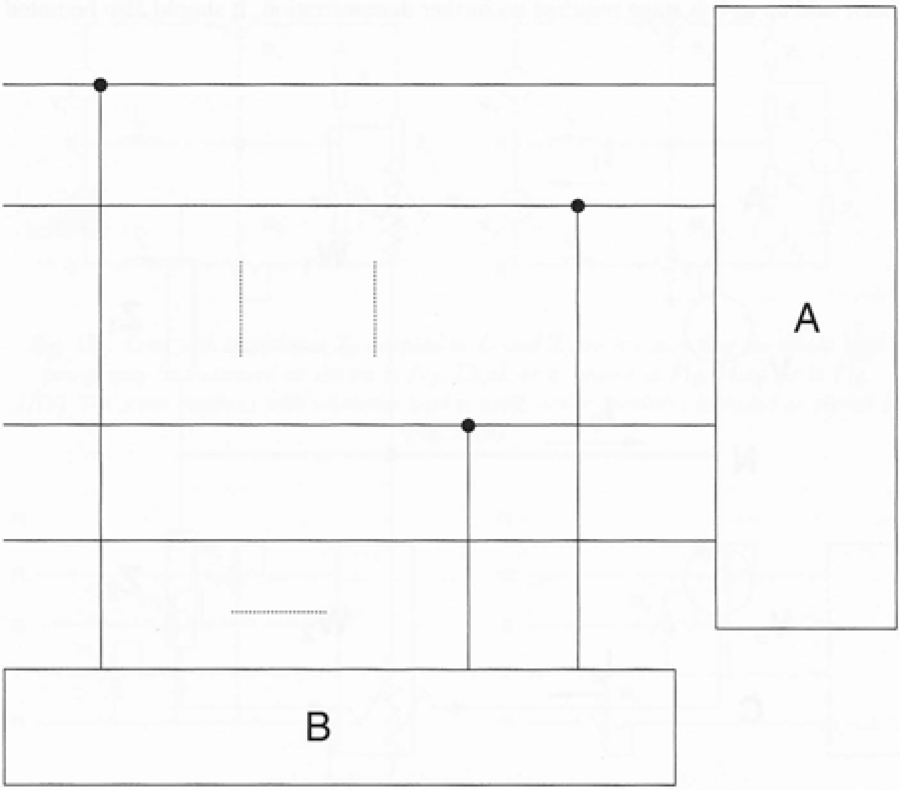

Fig. 9 *Multi-terminal networks in parallel.*

Power measurement of two-phase and three-phase loads

The preceding demonstration constitutes a unified theoretical and very simple basis for the measurement of polyphase loads. Figure 10 shows the usual manner in which to measure the power of a two-phase receiver with two wattmeters. Figure 11(a) shows an equivalent way, and Fig. 11(b) the procedure when three wattmeters are used. All of them are based on the first theorem demonstrated above in the section entitled 'Complex power absorbed by a multi-terminal network' which, in turn, is constructed on the preceding theoretical approach.

The last three connections are valid even though the receiver is composed by more than two impedances (Fig. 12(a)) or active loads (Fig. 12(b)), as the theorem refered to above is also legitimate whatever the multi-terminal network to which it is applied.

Figure 13(a) represents the usual way of measuring the power of a three-phase Y-load with three wattmeters; and Fig. 13(b) an equivalent one, which does not usually appear in texts.

The two wattmeters method now looks like an ordinary case of the more general procedure of measuring the power of a three-terminal network, irrespective of the configuration of the connections of the loads or the types of excitation. All that has been said up to this stage requires no further demonstration. It should also be noted

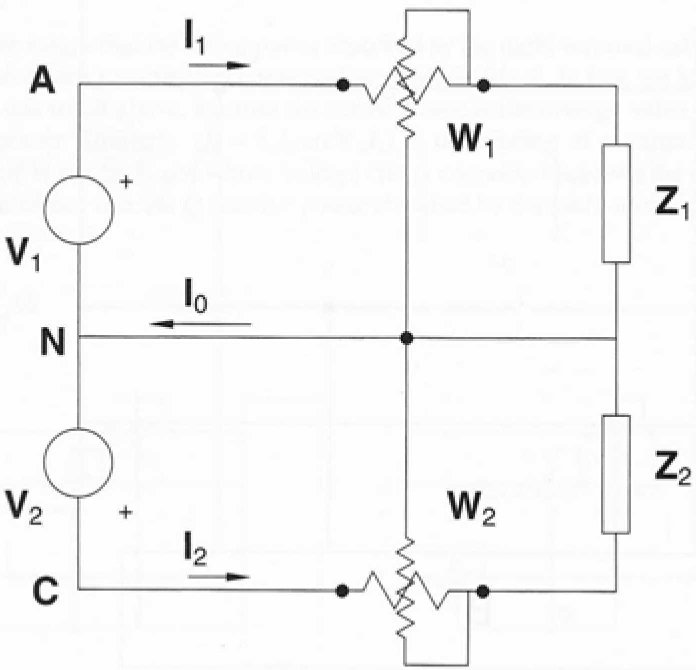

Fig. 10 *Usual method of measuring the power of a two-phase load.*

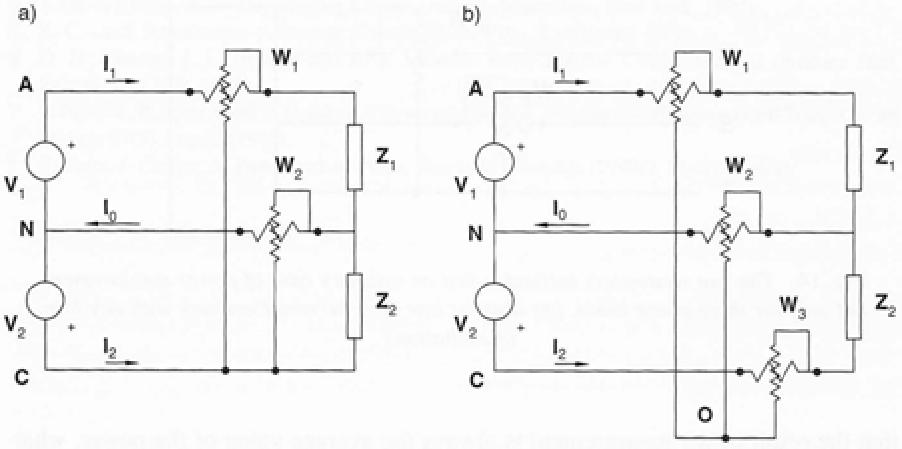

Fig. 11 *Two more possible manners to measure the power of a two-phase load.*

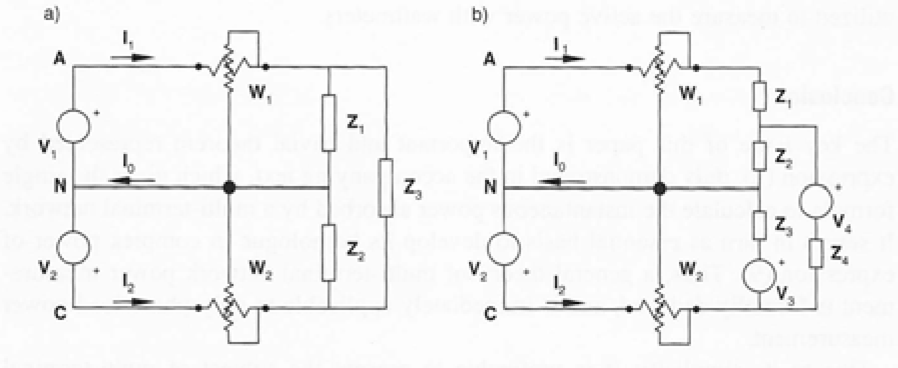

Fig. 12 *Even with impedance* $\mathbf{Z}_3$ *coupled to* $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *we are sure that the whole load power may be measured as shown in Fig. 12(a), or as shown in Fig. 11(a), or in Fig. 11(b). The same happens with whatever load is used, active receivers included as shown in Fig. 12(b).*

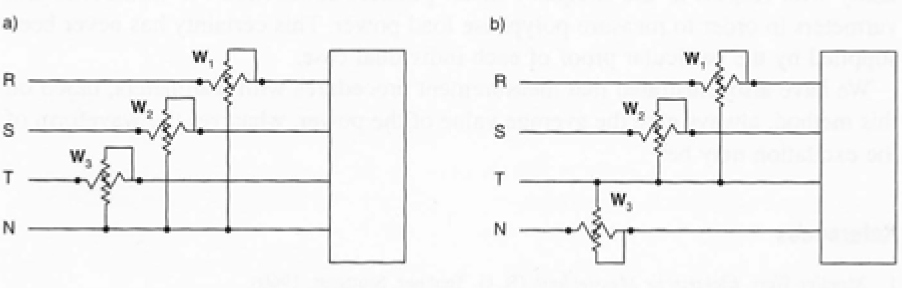

Fig. 13(a) *Represents the usual method of measuring power with the three wattmeters method; but the method in (b) is possible as well.*

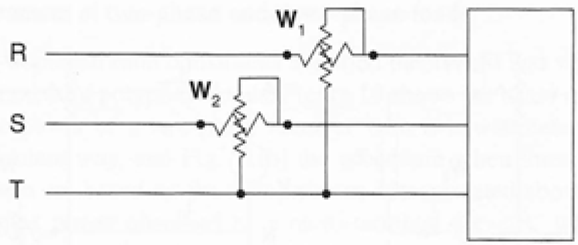

Fig. 14 *The two wattmeters method is just an ordinary case of power measurement, not only for three-phase loads, but also for any three-terminal network with any form of excitation.*

that the result of the measurement is always the average value of the power, whatever the waveform may be (Fig. 14).

As stated above, in the text accompanying expression (19), connections of varmeters used to measure reactive power in alternating current are just the same as those utilized to measure the active power with wattmeters.

Conclusion

The key issue of this paper is the important and trivial theorem represented by expression (2), duly demonstrated in the accompanying text, which gives the single formula to calculate the instantaneous power absorbed by a multi-terminal network. It serves in turn as essential basis to develop its homologue in complex power of expression (9). Thus, a general theory of multi-terminal network power measurement is formally deduced, and is immediately applicable to poly-phase load power measurement.

Due to its simplicity, it is preferable to present the subject of multi-terminal network power measurement in this unified manner. In particular, if the multi-terminal network is a poly-phase load, the need to justify each measurement method separately, as is usually done in textbooks, is eliminated, with consequent time and effort savings for the students. Furthermore, this presentation provides great certainty with respect to the adequacy of all possible connections of wattmeters and varmeters in order to measure polyphase load power. This certainty has never been supplied by the particular proof of each individual case.

We have also illustrated that measurement procedures with wattmeters, based on this method, always give the average value of the power, whatever the waveform of the excitation may be.

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